

On the characteristic polynomial of the Frobenius on étale cohomology

Andreas-Stephan Elsenhans* and Jörg Jahnel†

Abstract

Let X be a smooth projective variety of even dimension d over a finite field. Then we establish a restriction on the value at (-1) of the characteristic polynomial of the Frobenius on the middle-dimensional étale cohomology of X with coefficients in $\mathbb{Q}_l(d/2)$.

1 Introduction

Let X be a smooth projective variety over a finite field \mathbb{F}_q of characteristic $p > 0$. Then the geometric Frobenius Frob operates linearly on the l -adic cohomology vector spaces $H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$. The characteristic polynomial of Frob is independent of l as long as $l \neq p$ and has rational coefficients [De1, Théorème (1.6)].

By far not every polynomial may occur as the characteristic polynomial of the Frobenius on a smooth projective variety. The following conditions were established in the Grothendieck era.

1.1. Theorem (Deligne, Mazur, Ogus). — *Let X be a smooth projective variety over a finite field \mathbb{F}_q . For $i, j \in \mathbb{Z}$, denote by*

$$\Phi_j^{(i)} = T^N + a_1^{(i)} T^{N-1} + \cdots + a_{N-1}^{(i)} T + a_N^{(i)} \in \mathbb{Q}[T]$$

the characteristic polynomial of Frob on $H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$. Then

*Mathematisches Institut, Universität Bayreuth, Univ'straße 30, D-95440 Bayreuth, Germany, Stephan.Elsenhans@uni-bayreuth.de, Website: <http://www.staff.uni-bayreuth.de/~btm216>

†Département Mathematik, Universität Siegen, Walter-Flex-Str. 3, D-57068 Siegen, Germany, jahnel@mathematik.uni-siegen.de, Website: <http://www.uni-math.gwdg.de/jahnel>

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- a) every zero of $\Phi_j^{(i)}$ is of absolute value $q^{i/2-j}$.
b) If i is odd then all real zeroes of $\Phi_j^{(i)}$ are of even multiplicity.
c) For every l , prime to q , the zeroes of $\Phi_j^{(i)}$ are l -adic units, i.e., units in a suitable extension of \mathbb{Z}_l .
d) Put $h_{i-m,m} := \dim H^{i-m}(X, \Omega_X^m)$ and define the simple function $G^{(i)}: [0, N] \rightarrow \mathbb{R}$ by

$$G^{(i)}(t) = \begin{cases} 0 & \text{for } t \leq h_{i,0}, \\ n & \text{for } h_{i,0} + \dots + h_{i-n+1,n-1} < t \leq h_{i,0} + \dots + h_{i-n,n}. \end{cases}$$

Then, for $r = 1, \dots, N$, one has $a_r^{(i)} = 0$ or

$$\nu_q(a_r^{(i)}) \geq \int_0^r [G^{(i)}(t) - j] dt.$$

Here, ν_q is the non-archimedean valuation such that $\nu_q(q) = 1$.

Proof. a) This was proven by P. Deligne in [De1, Théorème (1.6)]. The assertion was first formulated by A. Weil as a part of his famous conjectures.

b) By the Hard Lefschetz theorem [De2, Théorème (4.1.1)] and Poincaré duality, there is a non-degenerate pairing

$$H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \times H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \rightarrow \mathbb{Q}_l(2j - i)$$

that is alternating and compatible with the operation of Frob. The assertion follows directly from this. Cf. the remarks after [De2, Corollaire (4.1.5)].

c) As Frob operates on $H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(j))$, the l -adic valuations of the eigenvalues are clearly non-negative. Poincaré duality implies the assertion, cf. [De1, (2.4)].

d) This statement was originally known as Katz's conjecture. The usual formulation is that the Newton polygon of $\Phi_0^{(i)}$ lies above the Hodge polygon of weight i . Proofs are due to B. Mazur [Maz2] and A. Ogus [BO, Theorem 8.39]. \square

1.2. Remarks. — i) Assertion a) immediately implies that $\Phi_j^{(i)} \in \mathbb{Q}[T]$ fulfills the functional equation

$$T^N \Phi(q^{i-2j}/T) = \pm q^{N(\frac{i}{2}-j)} \Phi(T)$$

for $N := \dim H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$. Indeed, on both sides, there are polynomials of degree N . They have the same zeroes as, with z , the number $\bar{z} = \frac{q^{i-2j}}{z}$ is a zero of Φ , too.

Further, by b), the plus sign always holds when i is odd. The statements c) and d) together show $\Phi_j^{(i)} \in \mathbb{Z}[T]$ when $j \leq 0$.

ii) For i even, we also have

$$\Phi_j^{(i)}(q^{i/2-j}) = 0. \quad (1)$$

Indeed, as X is projective, there is the $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ -invariant cycle given by the intersection of $i/2$ hyperplanes. The cycle map [SGA4 $\frac{1}{2}$, Cycle, Théorème 2.3.8.iii)] yields a non-trivial Galois invariant element of $H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(i/2))$.

1.3. Remark. — Consider the case that $i = 1$. Then $N = \dim H_{\text{ét}}^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ is always even [De2, Corollaire (4.1.5)]. On the other hand, let a polynomial $\Phi \in \mathbb{Z}[T]$ be given that is of even degree and fulfills assertions a), b), and c). Then, by the main theorem of T.Honda [Ho], there exists an abelian variety A such that the eigenvalues of Frob on $H_{\text{ét}}^1(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ are exactly the zeroes of Φ . One may enforce that the characteristic polynomial is a power of Φ , and, typically, Φ itself may be realized.

1.4. — We will show in this note that the same is not true in general for $i > 1$. In fact, for the characteristic polynomial of Frob on the middle cohomology of a variety of even dimension, we will establish a further condition, which is arithmetic in nature and independent of Theorem 1.1, as well as formula (1).

1.5. Theorem. — *Let X be a smooth projective variety of even dimension d over a finite field \mathbb{F}_q of characteristic p and $\Phi = \Phi_{d/2}^{(d)} \in \mathbb{Q}[T]$ be the characteristic polynomial of Frob on $H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(d/2))$. Put $N := \deg \Phi$.*

Then $(-2)^N \Phi(-1)$ is a square or p times a square in \mathbb{Q} .

1.6. Remark. — For X a surface, this result may be deduced from the Artin-Tate conjecture. Cf. Proposition 4.3, below.

1.7. — The correct exponent of p , may, at least for $p \neq 2$, be described as follows.

Definition. We put

$$e(X) := \sum_{m=0}^{\frac{d}{2}-1} \left(\frac{d}{2} - m\right) h'_{d-m,m},$$

for $(h'_{d-m,m})_m$ the abstract Hodge numbers of X in degree d [Maz1, Section 4].

1.8. Remarks. — a) Recall that the abstract Hodge numbers are defined as follows. The crystalline cohomology groups $H^i(X/W)$ are finitely generated W -modules, for $W := W(\mathbb{F}_q)$ the Witt ring. They are acted upon by the absolute Frobenius \mathbf{F} , the corresponding map is only \mathbf{F} -linear [Ch, Exposé I, 2.3.5].

Put $H := H^i(X/W)/\text{tors}$. Then, as $\mathbf{F}: H \rightarrow H$ is injective, $H/\mathbf{F}H$ is a W -module of finite length. By the classical invariant factor theorem, there is a unique

sequence of integers such that $H/\mathbf{F}H \cong \bigoplus_{m \geq 0} (W/p^m W)^{h'_{i-m,m}}$. Finally, one defines $h'_{i,0} := \mathrm{rk}_W H - \sum_{m \geq 0} h'_{i-m,m}$.

Observe that $e(\bar{X})$ is a geometric quantity. It depends only on the base extension $X_{\mathbb{F}_q}$.

b) Suppose that X is such that all $H^i(X/W)$ are torsion-free and that the conjugate spectral sequence $E_2^{j,m} := H^j(X, \mathcal{H}^m(\Omega_{X/\mathbb{F}_q}^\bullet)) \implies H_{\mathrm{dR}}^i(X)$ degenerates at E_2 . Then $h'_{i-m,m} = h_{i-m,m} (= \dim H^{i-m}(X, \Omega_X^m))$ [BO, Lemma 8.32].

For complete intersections, both assumptions hold ([SGA7, Exposé XI, Théorème 1.5] together with [BO, Lemma 8.27.2]). Further, the second assumption is automatically fulfilled when $\dim X \leq p$ and X lifts to W ([DI, Corollaire 2.4] and [BO, Lemma 8.27.2]).

c) Suppose that X is of Hodge-Witt type in degree d , i.e., that the Serre cohomology groups $H^j(X, W\Omega_X^m)$ [Se] are finitely generated W -modules for $j + m = d$. The assertion of Theorem 1.9 below may then be formulated entirely in terms of the characteristic polynomial Φ . In fact, denote the zeroes of Φ by z_1, \dots, z_N . Then

$$e(X) = - \sum_{\nu_q(z_i) < 0} \nu_q(z_i)$$

[Corollary 2.15]. This case includes all varieties that are ordinary in degree d [IR, Définition IV.4.12].

1.9. Theorem. — *Let X be a smooth projective variety of even dimension d over a finite field \mathbb{F}_q of characteristic $p \neq 2$ and $\Phi = \Phi_{d/2}^{(d)} \in \mathbb{Q}[T]$ be the characteristic polynomial of Frob on $H_{\mathrm{ét}}^d(X_{\mathbb{F}_q}, \mathbb{Q}_l(d/2))$. Put $N := \deg \Phi$.*

Then $(-2)^N q^{e(X)} \Phi(-1)$ is a square in \mathbb{Q} .

1.10. Remark. — The assertions may easily be formulated for an arbitrary Tate twist. Theorem 1.9 then states that $(-2)^N q^{e(X) - N(\frac{d}{2} - j)} \Phi_j^{(d)}(-q^{\frac{d}{2} - j})$ is a square in \mathbb{Q} .

2 The proof

2.1. Notation. — For R an integral domain, K its quotient field, and H an R -module, we will write $H_K := H \otimes_R K$.

2.2. Lemma. — *Let R be a principal ideal domain, K its quotient field, and H a free R -module of finite rank. Suppose there is a symmetric K -bilinear pairing $\langle \cdot, \cdot \rangle : H_K \times H_K \rightarrow K$ such that the restriction $\langle \cdot, \cdot \rangle|_{H \times H} : H \times H \rightarrow R$ is a perfect pairing. Further, let a K -linear map $\sigma : H_K \rightarrow H_K$ be given that is orthogonal with respect to the pairing, i.e., $\langle \sigma(x), \sigma(y) \rangle = \langle x, y \rangle$ for every $x, y \in H_K$. Put*

$$B_0(H) := [H / (H \cap (1 - \sigma)H)]_{\mathrm{tors}}.$$

a) Then there is a non-degenerate, skew-symmetric R -bilinear pairing

$$(\cdot, \cdot): B_0(H) \times B_0(H) \rightarrow K/R.$$

b) Suppose $\text{char } K \neq 2$ and $\langle x, x \rangle \in 2R$ for every $x \in H \cap (1 - \sigma)H_K$. Then (\cdot, \cdot) is alternating. In particular, the length of $B_0(H)$ is even.

2.3. Remarks. — i) Observe that $x \in H$ represents an element of $B_0(H)$ if and only if $x \in H \cap (1 - \sigma)H_K$.

ii) For $x, y \in H_K$ arbitrary, one has $\langle (1 - \sigma)x, \sigma y \rangle = -\langle x, (1 - \sigma)y \rangle$. In particular, as $\sigma: H_K \rightarrow H_K$ is bijective, $x \in \ker(1 - \sigma)$ if and only if $x \in ((1 - \sigma)H_K)^\perp$. I.e., $(1 - \sigma)H_K$ is exactly the set of all elements perpendicular to the eigenspace $H_{K,1}$. This fact is rather obvious, let us nevertheless emphasize that it is true whether σ is semisimple or not.

2.4. Proof of Lemma 2.2. — a) *Definition.* The pairing is defined as follows. For $a, b \in B_0(H)$, choose representatives $x, y \in H$. Let $y' \in H_K$ be such that $y = (1 - \sigma)y'$. Then $(a, b) := \langle x, y' \rangle \text{ mod } R$.

Well-definedness. For two representatives $x_1, x_2 \in H$, we have $x_1 - x_2 = (1 - \sigma)v$ for some $v \in H$. Thus, $\langle x_1 - x_2, y' \rangle = -\langle \sigma v, (1 - \sigma)y' \rangle \in R$, as both sides are in H . On the other hand, for two representatives $y'_1, y'_2 \in H_K$, we have $(1 - \sigma)(y'_1 - y'_2) = (1 - \sigma)w \in H$ for a suitable $w \in H$. Therefore, $\langle x, y'_1 - y'_2 \rangle = \langle x, w \rangle + \langle x, k \rangle$ for some $k \in H_{K,1}$. The first summand is in R , as both sides are elements of H . The second summand vanishes, since $x \in (1 - \sigma)H_K$. It is clear that (\cdot, \cdot) is R -bilinear.

Non-degeneracy. For $0 \neq b \in B_0(H)$, one has a representative y and some $y' \in H_K$ such that $y = (1 - \sigma)y'$. As $y \notin (1 - \sigma)H$, we see $y' \notin H + H_{K,1}$. The goal is to find some $x \in H \cap (1 - \sigma)H_K$ such that $\langle x, y' \rangle \notin R$.

For this, we observe that the perfect pairing induces an isomorphism $H_K \xrightarrow{\cong} \text{Hom}(H, K)$. Under this map, $H_{K,1} \cong \text{Hom}(H/[H \cap (1 - \sigma)H_K], K)$. Further,

$$H + H_{K,1} \cong \{\alpha \in \text{Hom}(H, K) \mid \alpha(H \cap (1 - \sigma)H_K) \subseteq R\}.$$

Indeed, as $H \cong \text{Hom}(H, R)$, the inclusion “ \subseteq ” is obvious. The other inclusion follows from the fact that $H \cap (1 - \sigma)H_K$ is a direct summand of H . The homomorphism $\alpha|_{H \cap (1 - \sigma)H_K}: H \cap (1 - \sigma)H_K \rightarrow R$ may thus be extended to a homomorphism $\alpha': H \rightarrow R$, corresponding to an element of H . The difference $\alpha - \alpha'$ vanishes on $H \cap (1 - \sigma)H_K$, hence is defined by an element of $H_{K,1}$.

Now, as $y' \notin H + H_{K,1}$, the corresponding homomorphism does not send $H \cap (1 - \sigma)H_K$ to R . I.e., there is some $x \in H \cap (1 - \sigma)H_K$, not mapped to R . This is exactly our claim.

Skew-symmetry. Let $a, b \in B_0(H)$ and choose representatives $x, y \in H$. There are $x', y' \in H_K$ such that $x = (1 - \sigma)x'$ and $y = (1 - \sigma)y'$.

Then $\langle x, y' \rangle + \langle x', y \rangle = \langle x' - \sigma x', y' \rangle + \langle x', y' - \sigma y' \rangle = \langle x' - \sigma x', y' - \sigma y' \rangle \in R$, again due to the fact that both sides are elements of H .

b) For $a \in B_0(H)$ choose a representative $x \in H$ and $x' \in H_K$ such that $x = (1 - \sigma)x'$. Then

$$2\langle x, x' \rangle = [\langle x', x' \rangle - \langle \sigma x', x' \rangle] + [\langle \sigma x', \sigma x' \rangle - \langle x', \sigma x' \rangle] = \langle x' - \sigma x', x' - \sigma x' \rangle \in 2R,$$

hence $\langle a, a \rangle = 0$. \square

2.5. Remarks. — i) If 2 is a unit in R then the assumption of b) is automatically fulfilled.

ii) When $R = \mathbb{Z}_l$, $\sigma(H) \subseteq H$, and σ is semisimple at 1, this result was proven by Y. Zarhin in [Zar, 3.3 and Lemma 3.4.1]. It is implicitly contained in the work of J.W.S. Cassels [Ca1].

iii) Most of our applications will be based on the following corollary.

2.6. Corollary. — *Let (R, ν) be a discrete valuation ring of characteristic $\neq 2$, K its quotient field, and H a free R -module of finite rank. Suppose there is a symmetric K -bilinear pairing $\langle \cdot, \cdot \rangle: H_K \times H_K \rightarrow K$ such that the restriction $\langle \cdot, \cdot \rangle|_{H \times H}: H \times H \rightarrow R$ is a perfect pairing. Further, let a K -linear map $\sigma: H_K \rightarrow H_K$ be given that is orthogonal with respect to the pairing and such that 1 is not among its eigenvalues.*

a) *Then $\ell(\sigma(H) + H/H) + \nu(\det(1 - \sigma))$ is even.*

b) *In particular, if $\sigma(H) \subseteq H$ then $\nu(\det(1 - \sigma))$ is even.*

Proof. a) All the assumptions of Lemma 2.2.b) are fulfilled. Further, as 1 is not an eigenvalue of σ , $H/[H \cap (1 - \sigma)H]$ is purely torsion. Thus, we have that $\ell(H/[H \cap (1 - \sigma)H])$ is even. Finally,

$$H/[H \cap (1 - \sigma)H] \cong [H + (1 - \sigma)H]/(1 - \sigma)H = [\sigma(H) + H]/(1 - \sigma)H$$

and $\nu(\det(1 - \sigma)) = \ell(M/(1 - \sigma)H) - \ell(M/H)$, for example for $M := \sigma(H) + H$.

b) is an immediate consequence of a). \square

2.7. — In order to illustrate the strength of Corollary 2.6, let us show an application to modules of rank two, the smallest non-trivial case. The fact obtained belongs to the not-so-well-known results on real quadratic number fields. Cf. [Zag, p. 118].

Corollary. *Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field and $\varepsilon \in K$ a unit of norm $(+1)$. Then $N(1 - \varepsilon) = 2 - \text{tr}(\varepsilon)$ is a product of some primes dividing the discriminant, a perfect square, and, possibly, a factor 2 and a minus sign.*

Proof. As K is a quadratic number field, the norm $N: \mathcal{O}_K \rightarrow \mathbb{Z}$ is a quadratic form. The multiplication map $\cdot \varepsilon: \mathcal{O}_K \rightarrow \mathcal{O}_K$ is compatible with this form and, therefore,

orthogonal with respect to the symmetric, bilinear form $\langle \cdot, \cdot \rangle: \mathcal{O}_K \times \mathcal{O}_K \rightarrow \mathbb{Z}$ associated to N .

The same is true for the corresponding \mathbb{Z}_l -valued pairings between the l -adic completions of \mathcal{O}_K , for l any prime number. As these pairings are perfect as long as l does not divide the discriminant of K , the assertion follows from Corollary 2.6.b). \square

2.8. Proof of Theorem 1.5. — We clearly have that $(-2)^N \Phi(-1) \in \mathbb{Q}$. Further, we may assume that (-1) is not among the zeroes of Φ as, otherwise, the assertion is true, trivially.

Then $(-2)^N \Phi(-1) > 0$. Indeed, as $\Phi \in \mathbb{R}[T]$ and there is no real zero different from 1,

$$(-2)^N \Phi(-1) = 2^N (1 + z_1) \cdot \dots \cdot (1 + z_N)$$

is the product of several factors of the form $z\bar{z} = |z|^2$ for $z \in \mathbb{C}$ and some factors that are equal to 2. To prove the assertion, we will show that $(-2)^N \Phi(-1)$ is of even l -adic valuation for every prime number $l \neq p$.

Put $H := H_{\text{ét}}^d(X_{\overline{\mathbb{F}_q}}, \mathbb{Z}_l(d/2))/\text{tors}$. By Poincaré duality ([SGA4, Exp. XVIII, formule (3.2.6.2)], cf. [Sp, Chap. 6, Sec. 2, Theorem 18 and Chap. 5, Sec. 5, Theorem 3]), the bilinear pairing

$$\langle \cdot, \cdot \rangle: H \times H \longrightarrow H_{\text{ét}}^{2d}(X_{\overline{\mathbb{F}_q}}, \mathbb{Z}_l(d)) \xrightarrow{\cong} \mathbb{Z}_l,$$

given by cup product and trace map, is perfect. As d is even, it is symmetric, too. The operation of Frob on H is orthogonal with respect to this pairing.

First case. $l \neq 2$.

The operation of $(-\text{Frob})$ is orthogonal with respect to the pairing, too. As 1 is not among its eigenvalues, Corollary 2.6.b) shows that $\nu_l(\det(1 + \text{Frob})) = \nu_l(\Phi(-1))$ is even.

Second case. $l = 2$.

Here, the argument is a bit more involved. First, we note that, by Lemma 2.9, there is a Frob-invariant element $\omega \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}_q}}, \mathbb{Z}_2(d/2))$ such that, for all $x \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}_q}}, \mathbb{Z}_2(d/2))$, one has $\langle \omega, x \rangle + \langle x, x \rangle \in 2\mathbb{Z}_2$.

For $x \in (1 - \text{Frob})H_{\mathbb{Q}_2}$, the fact that $\omega \in H_{K,1}$ implies $\langle \omega, x \rangle = 0$. Hence, $\langle x, x \rangle \in 2\mathbb{Z}_2$ for $x \in H \cap (1 - \text{Frob})H_{\mathbb{Q}_2}$.

According to Lemma 2.2.b), $[H/(1 - \text{Frob})H]_{\text{tors}}$ is of even length. An application to $X_{\overline{\mathbb{F}_q}}$ shows that $[H/(1 - \text{Frob}^2)H]_{\text{tors}}$ is of even length, too. Lemma 2.10 now yields the assertion. \square

2.9. Lemma. — *Let X be a smooth projective variety of even dimension d over a finite field \mathbb{F}_q of characteristic $\neq 2$. Then there is a Frob-invariant element*

$\omega \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$ such that

$$\langle \omega, x \rangle + \langle x, x \rangle \in 2\mathbb{Z}_2$$

for every $x \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$.

Proof. Denote by $\varepsilon: H_{\text{ét}}^{2*}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(*)) \rightarrow H_{\text{ét}}^{2*}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/2\mathbb{Z})$ the reduction map. Further, let $\nu_d \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/2\mathbb{Z})$ be the d -th Wu class of X [Ur, p. 578]. According to its very definition,

$$\nu_d \cup x + x \cup x = \text{Sq}^d(x) + \text{Sq}^d(x) = 0$$

for every $x \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/2\mathbb{Z})$ [Ur, Prop. 2.2.(2)]. Thus, a Frob-invariant element $\omega \in H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$ such that $\varepsilon(\omega) = \nu_d$ will serve our purposes.

In order to construct such a cohomology class, note first that the tangent bundle \mathcal{T}_X is defined over the base field. Hence, the Chern classes $c_i := c_i(\mathcal{T}_X) \in H_{\text{ét}}^{2i}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(i))$ are Frob-invariant. Further, for their Steenrod squares, there are the formulas

$$\text{Sq}^{2j}(\varepsilon(c_i)) = \varepsilon(c_j)\varepsilon(c_i) + \binom{2j-2i}{2}\varepsilon(c_{j-1})\varepsilon(c_{i+1}) + \cdots + \binom{2j-2i}{2j}\varepsilon(c_0)\varepsilon(c_{i+j}).$$

Indeed, these follow in a purely formal manner from the definitions of Chern classes and Steenrod squares, cf. [MS, Problem 8-A].

Finally, for every $k \in \mathbb{N}_0$, there is the formula of Wu [Ur, Proposition 0.5],

$$\text{Sq}^0(\nu_{2k}) + \text{Sq}^2(\nu_{2k-2}) + \cdots + \text{Sq}^{2k}(\nu_0) = \varepsilon(c_k).$$

As $\text{Sq}^0 = \text{id}$, a simple induction argument shows that, for every $k \in \mathbb{N}_0$, there is a polynomial expression for ν_{2k} in terms of $\varepsilon(c_0), \varepsilon(c_1), \dots, \varepsilon(c_k)$. In particular, ν_d has a Frob-invariant lift to $H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$. \square

2.10. Lemma. — *Let (R, ν) be a normalized discrete valuation ring of characteristic 0, K its quotient field, and H a free R -module of finite rank N , equipped with a non-degenerate, symmetric K -bilinear pairing $\langle \cdot, \cdot \rangle: H_K \times H_K \rightarrow K$. Further, let a K -linear map $\sigma: H_K \rightarrow H_K$ be given that is orthogonal with respect to the pairing and does not have the eigenvalue (-1) .*

In this situation, if each of the modules $[\sigma(H) + H]/H$, $[\sigma^2(H) + H]/H$, $[H/H \cap (1 - \sigma)H]_{\text{tors}}$, and $[H/H \cap (1 - \sigma^2)H]_{\text{tors}}$ is of even length then

$$\nu(\det(1 + \sigma)) \equiv N\nu(2) \pmod{2}.$$

Proof. *First step.* The general argument.

Since (-1) is not an eigenvalue of σ , the map $(1 + \sigma): H_K \rightarrow H_K$ is a bijection. In particular, $(1 - \sigma)H_K = (1 - \sigma^2)H_K$. Further, there is the commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
0 \twoheadrightarrow H \cap (1 - \sigma)H / H \cap (1 - \sigma)H \cap (1 - \sigma^2)H & \longrightarrow & H / H \cap (1 - \sigma)H \cap (1 - \sigma^2)H & \longrightarrow & H / H \cap (1 - \sigma)H & \twoheadrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow & H / H \cap (1 - \sigma)H_K \cap (1 - \sigma^2)H_K & = & H / H \cap (1 - \sigma)H_K & \twoheadrightarrow & 0.
\end{array}$$

As the vertical arrows are surjective, the 9-lemma yields exactness of

$$\begin{aligned}
0 \longrightarrow & H \cap (1 - \sigma)H / H \cap (1 - \sigma)H \cap (1 - \sigma^2)H \\
& \longrightarrow [H / H \cap (1 - \sigma)H \cap (1 - \sigma^2)H]_{\text{tors}} \longrightarrow [H / H \cap (1 - \sigma)H]_{\text{tors}} \longrightarrow 0.
\end{aligned}$$

To simplify notation, put $M := H \cap (1 - \sigma)H \cap (1 - \sigma^2)H$. We see that the length of $H \cap (1 - \sigma)H / M$ has the same parity as that of $[H / M]_{\text{tors}}$. Interchanging the roles of σ and σ^2 in the entire argument yields that the length of $H \cap (1 - \sigma^2)H / M$ has this parity, too.

Now observe $(1 - \sigma)H / H \cap (1 - \sigma)H \cong [\sigma(H) + H] / H$ and the analogous isomorphism for σ^2 . As $[\sigma(H) + H] / H$ and $[\sigma^2(H) + H] / H$ are supposed to be of even lengths, we may conclude that

$$\ell((1 - \sigma)H / M) \equiv \ell((1 - \sigma^2)H / M) \pmod{2},$$

as well. As $(1 - \sigma^2)H = (1 + \sigma)(1 - \sigma)H$, this shows that $\det(1 + \sigma)|_{(1 - \sigma)H_K}$ is of even valuation.

Second step. Jordan blocks.

According to C. Jordan, we have $H_K = \ker(1 - \sigma)^r + (1 - \sigma)H_K$ for r large. On $\ker(1 - \sigma)^r$, the Frobenius only has the eigenvalue 1. Hence,

$$\begin{aligned}
\nu(\det(1 + \sigma)) &\equiv [N - \dim(1 - \sigma)H_K]\nu(2) \pmod{2} \\
&= [\dim \ker(1 - \sigma)]\nu(2).
\end{aligned}$$

Finally, σ is orthogonal with respect to a non-degenerate bilinear form. In this situation, [Wa, Example 2.6.C.i.b)] (see also [St, §1.A1, Lemma 4.ii]) shows that

$$\dim \ker(1 - \sigma) \equiv \dim \ker(1 - \sigma)^r \pmod{2}$$

for $r \gg 0$. As (-1) is not an eigenvalue, the latter has the same parity as N . The assertion follows. \square

2.11. Remarks. — i) There is a conjecture of J.-P. Serre that the operation of Frob on l -adic cohomology is always semisimple. Then, as the eigenvalues come in pairs $\{z, \frac{1}{z}\}$, the congruence $\dim \ker(1 - \text{Frob}) \equiv N \pmod{2}$ is clear. Thus, conjecturally, the argument on the sizes of the Jordan blocks is not necessary in the application to l -adic cohomology.

ii) Suppose that $q = p^k$ for a prime $p \neq 2$ and let X be a surface such that the canonical sheaf $K \in \text{Pic}(X_{\overline{\mathbb{F}}_q})$ is divisible by 2. Then the case $l = 2$ of Theorem 1.5 may be treated directly.

Indeed, in this situation, Wu's formula [Ur, Proposition 2.1] implies that $\langle x, x \rangle \in 2\mathbb{Z}_2$ for every $x \in H$. Therefore, by Lemma 2.2.b), $H/(1 + \text{Frob})H$ is of even length. Further, the assumption enforces that K^2 is even. Hence, $N = \dim H_{\text{ét}}^2(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_2(1))$ is even, too, by Noether's formula [Bea, I.14].

2.12. Proof of Theorem 1.9. — Here, according to our assumption, we have $p \neq 2$. Again, we may assume without restriction that $\Phi(-1) \neq 0$. In view of Theorem 1.5, it will suffice to prove that $q^{e(X)}\Phi(-1)$ is of even p -adic valuation. Writing $q = p^k$, this means that $ke + \nu_p((1 + z_1) \cdots (1 + z_n))$ is even.

For this, let $W := W(\mathbb{F}_q)$ be the Witt ring and K its quotient field. The crystalline cohomology groups $H^d(X/W)$ are finitely generated W -modules, acted upon semi-linearly by the absolute Frobenius \mathbf{F} . The operation of \mathbf{F}^k is W -linear and such that its characteristic polynomial coincides with $\Phi_0^{(d)}$ [KM].

Again, $H := H^d(X/W)/\text{tors}$ is equipped with a natural perfect pairing [Ber, Ch. VII, Théorème 2.1.3]

$$\langle \cdot, \cdot \rangle : H \times H \longrightarrow W.$$

The Frobenius operation is, however, compatible with this pairing only in the sense that $\langle \mathbf{F}(x), \mathbf{F}(y) \rangle = p^d \langle x, y \rangle$ for $x, y \in H$ [Ch, Exposé II, Exemple 1.1.ii]. The map $\sigma := \mathbf{F}/p^{d/2} : H_K \rightarrow H_K$ is orthogonal with respect to the pairing $\langle \cdot, \cdot \rangle$.

The Dieudonné module H carries a rich structure [Man]. We will use only a small part of it. In fact, H is a free module as well over the Witt ring $W(\mathbb{F}_p)$ and σ is $W(\mathbb{F}_p)$ -linear. The eigenvalues of σ , as a $W(\mathbb{F}_p)$ -linear map, are all the k -th roots of the zeroes z_1, \dots, z_N of Φ . Further, σ is orthogonal with respect to the perfect pairing $\text{tr} \circ \langle \cdot, \cdot \rangle : H \times H \rightarrow W(\mathbb{F}_p)$. Let us now distinguish two cases.

First case. k is odd.

The operation of $(-\sigma)$ is orthogonal with respect to the pairing, too. Thus, Corollary 2.6 shows that $\ell_{W(\mathbb{F}_p)}(\sigma(H) + H/H) + \nu_{W(\mathbb{F}_p)}(\det(1 + \sigma))$ is even. By Lemma 2.14, the first summand is equal to ke . The second summand is the p -adic valuation of

$$\prod_{i=1}^N \prod_{r^k = z_i} (1 + r) = \prod_{i=1}^N (1 + z_i).$$

Second case. k is even.

Here, our task is to verify that $\nu_{W(\mathbb{F}_p)}(\det(1 + \sigma))$ is even. For this, note that, as σ is \mathbf{F} -linear, $\sigma(H)$ and $\sigma^2(H)$ are W -modules. In particular, $\ell_{W(\mathbb{F}_p)}(\sigma(H) + H/H)$ and $\ell_{W(\mathbb{F}_p)}(\sigma^2(H) + H/H)$ are even.

Further, Lemma 2.2.b) shows that $\ell_{W(\mathbb{F}_p)}([H/H \cap (1 - \sigma)H]_{\text{tors}})$ and $\ell_{W(\mathbb{F}_p)}([H/H \cap (1 - \sigma^2)H]_{\text{tors}})$ are even, too. The claim is now implied by Lemma 2.10. \square

2.13. Remark. — One might want to prove Theorem 1.9 in characteristic 2 along the same lines as Theorem 1.5. For this, one would need the theory of Steenrod squares and Wu classes, as well as Wu’s formula, for crystalline cohomology of varieties in characteristic 2. It seems however that such a theory is not yet available in the literature.

2.14. Lemma. — *Let X be a smooth projective variety of even dimension d over \mathbb{F}_q , $H := H^d(X/W)/\text{tors}$, and $\sigma = \mathbf{F}/p^{d/2}$. Then $\ell_W(\sigma(H) + H/H) = e$.*

Proof. First, observe that $\sigma(H)$, being the image of an \mathbf{F} -linear map, is indeed a W -module. Further, we have $H/\mathbf{F}H \cong \bigoplus_{m>0} (W/p^m W)^{h'_{d-m,m}}$. Hence, there is a basis of H such that, under the isomorphism $H \cong W^N$, one has $\mathbf{F}H \cong \bigoplus_{m \geq 0} p^m W^{h'_{d-m,m}}$. Therefore, $\sigma(H) \cong \bigoplus_{m \geq 0} p^{m-d/2} W^{h'_{d-m,m}}$. The assertion follows. \square

2.15. Corollary. — *Let X be a smooth projective variety of even dimension d over \mathbb{F}_q . Suppose that X is of Hodge-Witt type in degree d , i.e., that the Serre cohomology groups $H^j(X, W\Omega_X^m)$ are finitely generated W -modules for $j + m = d$. Then*

$$e = - \sum_{\nu_q(z_i) < 0} \nu_q(z_i).$$

Proof. By [IR, Théorème IV.4.5], we have $H^d(X/W) \cong \bigoplus_m H^{d-m,m}$ for $H^{d-m,m} := H^{d-m}(X, W\Omega_X^m)$. On $H^{d-m,m}$, \mathbf{F} operates as $p^m F$ for F the usual Frobenius on Serre cohomology. Thus, σ acts as $p^{m-d/2} F$. For $m \geq d/2$, this ensures that the corresponding summand is mapped to H .

Thus, assume that $m < d/2$. On Serre cohomology, there is a second operator, the Verschiebung V , such that $FV = p$. Hence $\sigma p^{d/2-m-1} V = \text{id}$, implying $H^{d-m,m} \otimes_W \text{Quot}(W) \supseteq \sigma(H^{d-m,m}) \supseteq H^{d-m,m}$. Lemma 2.14 shows that $e = -\nu_q(\det(\sigma)|_{\bigoplus_{m < d/2} H^{d-m,m}})$, which is equivalent to the assertion. \square

3 Algebro-geometric constructions

The classical conditions satisfied by the characteristic polynomials of Frob, as listed in Theorem 1.1, clearly have the property that they carry over from a variety X to its base extensions, its Albanese variety, and to direct products.

For the new condition, the same is not at all obvious. One might hope that applying Theorem 1.9 to one of these algebro-geometric constructions leads to another restriction for the characteristic polynomial of Frob on X itself. However, this does not happen. The reason is that for varieties obtained by base extension and for products, the assertion of Theorem 1.9 may be verified by elementary means.

3.1 Extending the base field

3.1. Notation. — Let k be an integer. For a polynomial $\Phi \in \mathbb{Q}[T]$ with zeroes z_1, \dots, z_N , let $\Phi^{(k)}$ be the monic polynomial with zeroes z_1^k, \dots, z_N^k .

3.2. Lemma. — Let $k \in \mathbb{Z}$. Assume that the polynomial $\Phi \in \mathbb{Q}[T]$ fulfills a functional equation

$$\Phi_1(1/T) = \pm \frac{1}{T^N} \Phi_1(T)$$

for $N := \deg \Phi$.

i) If k is even then $(-2)^N \Phi^{(k)}(-1)$ is a square in \mathbb{Q} .

ii) If $(-2)^N p^e \Phi(-1)$ is a square in \mathbb{Q} then $(-2)^N p^{ke} \Phi^{(k)}(-1)$ is a square in \mathbb{Q} , too.

Proof. The functional equation implies that, except for 1 and (-1) , the zeroes of Φ come in pairs z and $1/z$.

i) As factors $(T-1)$ neither change the assumption nor the assertion, it is sufficient to consider the case that $\Phi(-1) \neq 0$ and $\Phi(1) \neq 0$. Now observe the identity

$$(-1 - z^k)(-1 - 1/z^k) = (z^{k/2} + 1/z^{k/2})^2. \quad (2)$$

It shows that the product $\prod_i (-1 - z_i^k)$ is a perfect square. Indeed, the sums occurring on the right hand side of (2) form a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -invariant set.

ii) We may assume that k is odd. If $\Phi(-1) = 0$ then the assertion is true, trivially. Further, factors $(T-1)$ are irrelevant to the statements considered. Hence, we may restrict ourselves to $\Phi(-1) \neq 0$ and $\Phi(1) \neq 0$. We claim that $\Phi^{(k)}(-1)/\Phi(-1)$ is a perfect square.

For this, we calculate

$$\Phi^{(k)}(-1)/\Phi(-1) = \prod_i \frac{(-1 - z_i^k)}{(-1 - z_i)} = \prod_i \prod_{j=0}^{k-1} (1 + \zeta_k^j z_i). \quad (3)$$

As $(1 + \zeta_k^j z_i)(1 + \zeta_k^j \cdot 1/z_i)(1 + \zeta_k^{-j} z_i)(1 + \zeta_k^{-j} \cdot 1/z_i) = (z_i + 1/z_i + \zeta_k^j + \zeta_k^{-j})^2$, we see that the right hand side of (3) is the square of a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -invariant expression. \square

3.3. Remark. — Hence, when applying Theorem 1.9 to base extensions of a variety X over a finite field \mathbb{F}_{q^k} , no further restrictions appear on the characteristic polynomial of Frobenius on $H_{\text{ét}}^d(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l(d/2))$.

3.2 Direct products

3.4. Lemma. — Let p be a prime and d_1 and d_2 two integers that are either both even or both odd. Further, let $\Phi_1 \in \mathbb{Q}[T]$ and $\Phi_2 \in \mathbb{Q}[T]$ be polynomials of even degrees N_1 and N_2 that fulfill the functional equations

$$\Phi_1(p^{d_1}/T) = \frac{p^{d_1 N_1/2}}{T^{N_1}} \Phi_1(T) \quad \text{and} \quad \Phi_2(p^{d_2}/T) = \frac{p^{d_2 N_2/2}}{T^{N_2}} \Phi_2(T).$$

For $z_i^{(1)}$ the zeroes of Φ_1 and $z_j^{(2)}$ the zeroes of Φ_2 , let Φ be the monic polynomial with the zeroes $z_i^{(1)}z_j^{(2)}/p^{\frac{d_1+d_2}{2}}$.

i) If d_1 and d_2 are even then $(-2)^{N_1N_2}\Phi(-1)$ is a square in \mathbb{Q} .

ii) If d_1 and d_2 are odd then $(-2)^{N_1N_2}p^e\Phi(-1)$ is a square in \mathbb{Q} , for $e = \frac{N_1N_2}{4}$.

Proof. The assumption implies that the zeroes come in pairs with products p^{d_1} and p^{d_2} , respectively. For two such pairs, the corresponding four zeroes of Φ are

$$\frac{z_i^{(1)}}{p^{d_1/2}} \cdot \frac{z_j^{(2)}}{p^{d_2/2}}, \quad \frac{z_i^{(1)}}{p^{d_1/2}} \cdot \frac{p^{d_2/2}}{z_j^{(2)}}, \quad \frac{p^{d_1/2}}{z_i^{(1)}} \cdot \frac{z_j^{(2)}}{p^{d_2/2}}, \quad \text{and} \quad \frac{p^{d_1/2}}{z_i^{(1)}} \cdot \frac{p^{d_2/2}}{z_j^{(2)}}.$$

Now observe the identity

$$(-1 - z_1z_2)(-1 - z_1/z_2)(-1 - z_2/z_1)(-1 - 1/z_1z_2) = (z_1 + z_2 + 1/z_1 + 1/z_2)^2. \quad (4)$$

It shows that the product $\prod_{i,j}(-1 - z_i^{(1)}z_j^{(2)}/p^{\frac{d_1+d_2}{2}})$ is a square or p times a square. Indeed, the sums occurring on the right hand side of (4) form a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{p}))$ -invariant set.

When d_1 and d_2 are even, we actually have invariance under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This implies assertion i). When d_1 and d_2 are odd, the sums on the right hand side of (4) form a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{p}))$ -invariant set, but an automorphism sending \sqrt{p} to $(-\sqrt{p})$ changes the sign. Hence, the exponent of p is the number of such products of four occurring, that is $N_1N_2/4$. \square

3.5. Lemma. — Let $\Phi_1 \in \mathbb{Q}[T]$ and $\Phi_2 \in \mathbb{Q}[T]$ be polynomials of degrees N_1 and N_2 that fulfill the functional equations

$$\Phi_1(1/T) = (-1)^{k_1} \frac{1}{T^{N_1}} \Phi_1(T) \quad \text{and} \quad \Phi_2(1/T) = (-1)^{k_2} \frac{1}{T^{N_2}} \Phi_2(T)$$

for certain $k_1, k_2 \in \{0, 1\}$. For $z_i^{(1)}$ the zeroes of Φ_1 and $z_j^{(2)}$ the zeroes of Φ_2 , let Φ be the monic polynomial with the zeroes $z_i^{(1)}z_j^{(2)}$.

Further, fix a prime p . If $k_1 = 1$ then suppose $(-2)^{N_2}p^{e_2}\Phi_2(-1)$ to be a square in \mathbb{Q} , for some e_2 , and vice versa. Then $(-2)^{N_1N_2}p^e\Phi(-1)$ is a square in \mathbb{Q} for $e = k_1e_2 + k_2e_1$.

Proof. If $\Phi_1(-1) = 0$ or $\Phi_2(-1) = 0$ then the assertion is true, trivially. Thus, assume that this is not the case.

Except possibly for those being equal to 1, the zeroes of Φ_1 and Φ_2 come in pairs z and $1/z$. Hence, we may write $\Phi_1(T) = (T-1)^{k_1}\Phi'_1(T)$ and $\Phi_2(T) = (T-1)^{k_2}\Phi'_2(T)$ for Φ_1, Φ_2 fulfilling the analogous functional equations with the plus sign. Consequently, there is a decomposition

$$\Phi(T) = (T-1)^{k_1k_2}\Phi'_1(T)^{k_2}\Phi'_2(T)^{k_1}\Phi'(T),$$

where $(-2)^{\deg \Phi'_1 \deg \Phi'_2}\Phi'(-1)$ is a square in \mathbb{Q} , according to Lemma 3.4.i). The assertion follows. \square

3.6. Proposition. — Let X_1 and X_2 be smooth projective varieties of dimensions d_1 and d_2 over a finite field \mathbb{F}_q of characteristic p . Put $N_1 := \dim H_{\text{ét}}^{d_1}(X_1, \overline{\mathbb{F}_q}, \mathbb{Q}_l)$ and $N_2 := \dim H_{\text{ét}}^{d_2}(X_2, \overline{\mathbb{F}_q}, \mathbb{Q}_l)$.

Assume that $d_1 \equiv d_2 \pmod{2}$ and denote by Φ be the characteristic polynomial of Frobenius on $H_{\text{ét}}^{d_1+d_2}((X_1 \times X_2)_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l(\frac{d_1+d_2}{2}))$.

i) If d_1 and d_2 are both odd then $(-2)^{N_1 N_2} p^e \Phi(-1)$ is a square in \mathbb{Q} , for the exponent $e = \frac{N_1 N_2}{4}$.

ii) Assume that d_1 and d_2 are both even. Let Φ_1 and Φ_2 be the characteristic polynomials of Frobenius on $H_{\text{ét}}^{d_1}(X_1, \overline{\mathbb{F}_q}, \mathbb{Q}_l(d_1/2))$ and $H_{\text{ét}}^{d_2}(X_2, \overline{\mathbb{F}_q}, \mathbb{Q}_l(d_2/2))$. If N_1 is odd then suppose $(-2)^{N_2} p^{e_2} \Phi_2(-1)$ to be a square in \mathbb{Q} , for some e_2 , and vice versa.

Then $(-2)^{N_1 N_2} p^e \Phi(-1)$ is a square in \mathbb{Q} , for the exponent $N_1 e_2 + N_2 e_1$.

Proof. In both cases, $d := \dim X + \dim Y$ is even. By the Künneth formula, the characteristic polynomial $\Phi_{X \times Y}$ splits into factors $F^{i,j}$ for $i + j = d$. Poincaré duality implies that $F^{i,j} = F^{2d_1-i, 2d_2-j}$. Hence, $\Phi_{X \times Y}$ is the product of a perfect square and the polynomial F^{d_1, d_2} .

i) For F^{d_1, d_2} , the assertion follows immediately from Lemma 3.4.ii).

ii) Here, we have $\Phi_1(1) = \Phi_2(1) = 0$. We may assume that $\Phi_1(-1) \neq 0$ and $\Phi_2(-1) \neq 0$ as, otherwise, the assertion is true, trivially. But then $N_1 \equiv k_1 \pmod{2}$ and $N_2 \equiv k_2 \pmod{2}$. The assertion for F^{d_1, d_2} follows from Lemma 3.5. \square

3.7. — The invariant $e(X_1 \times X_2)$ of the direct product behaves exactly in the same way as the exponent e , obtained by elementary means in the Proposition above.

Lemma. Let X_1 and X_2 be smooth projective varieties of dimensions d_1 and d_2 over a finite field \mathbb{F}_q of characteristic p . Put $N_1 := \dim H_{\text{ét}}^{d_1}(X_1, \overline{\mathbb{F}_q}, \mathbb{Q}_l)$ and $N_2 := \dim H_{\text{ét}}^{d_2}(X_2, \overline{\mathbb{F}_q}, \mathbb{Q}_l)$.

i) If d_1 and d_2 are both odd then $e(X_1 \times X_2) \equiv \frac{N_1 N_2}{4} \pmod{2}$.

ii) If d_1 and d_2 are both even then $e(X_1 \times X_2) \equiv N_1 e(X_2) + N_2 e(X_1) \pmod{2}$.

Proof. *First step.* Preparations.

We first observe that, for X any smooth projective variety of dimension d , the abstract Hodge numbers of X fulfill the Serre type relation $h'_{n,m} = h'_{d-n, d-m}$.

Indeed, the equation $\langle \mathbf{F}x, \mathbf{F}y \rangle = p^d \langle x, y \rangle$ for all $x \in H^{n+m}(X/W)/\text{tors}$ and $y \in H^{2d-n-m}(X/W)/\text{tors}$ implies $\mathbf{F}^* = p^d \mathbf{F}^{-1}$ for the operator adjoint to \mathbf{F} . As the pairing is perfect, \mathbf{F} and \mathbf{F}^* have the same invariant factors. However, every factor p^m of \mathbf{F} yields a factor p^{d-m} of $p^d \mathbf{F}^{-1}$.

We shall use the notation $h'_{i,\text{ev}} := h'_{i,0} + h'_{i-2,2} + h'_{i-4,4} + \dots$ and $h'_{i,\text{od}} := h'_{i-1,1} + h'_{i-3,3} + h'_{i-5,5} + \dots$. Clearly, one has

$$h'_{i,\text{ev}} + h'_{i,\text{od}} = h_i := \text{rk } H^i(X/W).$$

Furthermore, the determinantal relation $\sum_k k \cdot h'_{i-k,k} = \frac{i}{2} h_i$ yields

$$\begin{aligned} h'_{i,\text{od}} &\equiv \frac{i}{2} h_i \pmod{2}, \\ h'_{i,\text{ev}} &\equiv (1 - \frac{i}{2}) h_i \pmod{2}. \end{aligned}$$

Note that we do not have Hodge symmetry, $h'_{n,m} = h'_{m,n}$, at our disposal.

Second step. The Künneth formula.

We claim that, when applying the Künneth formula to $X_1 \times X_2$, all terms, except for those coming from the middle cohomologies, yield an even contribution to $e(X_1 \times X_2)$.

Let us consider the case that $\frac{d_1+d_2}{2}$ is odd, first. Then, for $n > 0$ arbitrary, the Serre type relation shows that the abstract Hodge numbers corresponding to

$$H^{d_1-n}(X_1/W) \times H^{d_2+n}(X_2/W) \oplus H^{d_1+n}(X_1/W) \times H^{d_2-n}(X_2/W)$$

are symmetric. Hence, modulo 2, the contribution to $e(X_1 \times X_2)$ of this summand is

$$\begin{aligned} &\frac{h'^{(X_1)}_{d_1-n,\text{ev}} h'^{(X_2)}_{d_2+n,\text{ev}} + h'^{(X_1)}_{d_1-n,\text{od}} h'^{(X_2)}_{d_2+n,\text{od}} + h'^{(X_1)}_{d_1+n,\text{ev}} h'^{(X_2)}_{d_2-n,\text{ev}} + h'^{(X_1)}_{d_1+n,\text{od}} h'^{(X_2)}_{d_2-n,\text{od}}}{2} \\ &= h'^{(X_1)}_{d_1-n,\text{ev}} h'^{(X_2)}_{d_2+n,\text{ev}} + h'^{(X_1)}_{d_1-n,\text{od}} h'^{(X_2)}_{d_2+n,\text{od}} \quad (\text{using the Serre relation}) \\ &\equiv (1 - \frac{d_1-n}{2}) h'^{(X_1)}_{d_1-n} (1 - \frac{d_2+n}{2}) h'^{(X_2)}_{d_2+n} + \frac{d_1-n}{2} h'^{(X_1)}_{d_1-n} \frac{d_2+n}{2} h'^{(X_2)}_{d_2+n} \pmod{2} \\ &\equiv (1 - \frac{d_1+d_2}{2} + \frac{(d_1-n)(d_2+n)}{2}) h'^{(X_1)}_{d_1-n} h'^{(X_2)}_{d_2+n} \pmod{2}. \end{aligned}$$

As $\frac{d_1+d_2}{2}$ is odd, the last formula is equivalent to $\frac{1}{2}(d_1-n) h'^{(X_1)}_{d_1-n} (d_2+n) h'^{(X_2)}_{d_2+n}$. But $ih'_i^{(X)}$ is always even.

If $\frac{d_1+d_2}{2}$ is even then we need $h'^{(X_1)}_{d_1-n,\text{ev}} h'^{(X_2)}_{d_2+n,\text{od}} + h'^{(X_1)}_{d_1-n,\text{od}} h'^{(X_2)}_{d_2+n,\text{ev}}$ instead. An analogous calculation leads to the same result.

Third step. The middle-dimensional cohomologies.

The result of the second step shows

$$\begin{aligned} e(X_1 \times X_2) &\equiv \sum_{\substack{m_1+m_2 < \frac{d_1+d_2}{2}}} (\frac{d_1+d_2}{2} - m_1 - m_2) h'^{(X_1)}_{d_1-m_1,m_1} h'^{(X_2)}_{d_2-m_2,m_2} \pmod{2} \\ &\equiv \sum_{\substack{m_1+m_2 < \frac{d_1+d_2}{2} \\ m_1+m_2 - \frac{d_1+d_2}{2} \equiv 1 \pmod{2}}} h'^{(X_1)}_{d_1-m_1,m_1} h'^{(X_2)}_{d_2-m_2,m_2} \pmod{2}. \end{aligned}$$

i) Among the four pairs (m_1, m_2) , (d_1-m_1, m_2) , (m_1, d_2-m_2) , and (d_1-m_1, d_2-m_2) , which cause equal products due to Serre, exactly two fulfill the inequality. Only one of these two has the right parity. Thus, we actually consider the term

$$\frac{1}{4} \sum_{m_1, m_2} h'^{(X_1)}_{d_1-m_1,m_1} h'^{(X_2)}_{d_2-m_2,m_2} = \frac{1}{4} \sum_{m_1} h'^{(X_1)}_{d_1-m_1,m_1} \sum_{m_2} h'^{(X_2)}_{d_2-m_2,m_2} = \frac{N_1 N_2}{4}.$$

ii) Here, in a set of four as above, the parities are all the same. If it is even then no product will contribute to the sum. If it is odd then exactly two of the four will do. This causes an even sum.

It remains to consider the pairs with $m_1 = \frac{d_1}{2}$ or $m_2 = \frac{d_2}{2}$, as they do not belong to any of the sets of four. This shows

$$\begin{aligned} e(X_1 \times X_2) &\equiv h'_{\frac{d_1}{2}, \frac{d_1}{2}}(X_1) \cdot \sum_{\substack{m_2 < \frac{d_2}{2} \\ m_2 - \frac{d_2}{2} \equiv 1 \pmod{2}}} h'_{d_2 - m_2, m_2}(X_2) + h'_{\frac{d_2}{2}, \frac{d_2}{2}}(X_2) \cdot \sum_{\substack{m_1 < \frac{d_1}{2} \\ m_1 - \frac{d_1}{2} \equiv 1 \pmod{2}}} h'_{d_1 - m_1, m_1}(X_1) \pmod{2} \\ &\equiv h'_{\frac{d_1}{2}, \frac{d_1}{2}}(X_1) \cdot e(X_2) + h'_{\frac{d_2}{2}, \frac{d_2}{2}}(X_2) \cdot e(X_1) \pmod{2}. \end{aligned}$$

Finally, for every smooth projective variety X of even dimension d , one has $h'_{\frac{d}{2}, \frac{d}{2}}(X) \equiv \text{rk}_W H^d(X/W) \pmod{2}$, due to the Serre type relation. \square

3.8. Remark. — Let X be a d -dimensional smooth projective variety over a finite field \mathbb{F}_{p^k} of characteristic $p \neq 2$. Then applying Theorem 1.9 to the self-product $X \times X$ or to $X \times Y$ for a fixed variety Y does not lead to restrictions on the characteristic polynomial of Frob on $H_{\text{ét}}^d(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l(d))$, different from those already known. This is implied by Proposition 3.6 together with Lemma 3.7.

3.9. Remark. — When applying Theorem 1.5 to the Albanese variety of X , no new restrictions result, either. We will not work this out explicitly as the arguments are similar to those given in the proof of Lemma 3.4.

3.3 An application to the odd-dimensional supersingular case

3.10. Proposition. — *Let X be a smooth projective variety of odd dimension d over a finite field \mathbb{F}_{p^k} for a prime $p \neq 2$ and k odd. Suppose that $\dim H_{\text{ét}}^d(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_l) \equiv 2 \pmod{4}$ and that all eigenvalues of Frob on that cohomology are of p -adic valuation $dk/2$. I.e., that the Newton polygon has constant slope $d/2$. Then $\pm\sqrt{-p^{dk}}$ are among the eigenvalues.*

Proof. Let C be a supersingular elliptic curve defined over \mathbb{F}_{p^k} . Such do exist by the work of M. Eichler [Ei], see also [Gr, Proposition 2.4, together with (1.10) and (1.11)]. Then all eigenvalues of Frob on

$$V := [H_{\text{ét}}^d(X, \mathbb{Q}_l) \times H_{\text{ét}}^1(C, \mathbb{Q}_l)](\frac{d+1}{2})$$

are p -adic units. As they are l -adic units for every prime $l \neq p$, too, they must be roots of unity [Ca2, Sec. 18, Lemma 2].

For Φ the characteristic polynomial of Frob on V , Lemma 3.4.ii) guarantees that $p\Phi(-1)$ is a perfect square. In view of Lemma 3.12 below, this is possible only

for $\Phi(-1) = 0$. Hence, (-1) is an eigenvalue of Frob on V . The assertion follows. \square

3.11. Remark. — In principle, the idea behind this proof is to apply Theorem 1.9 to $X \times C$. This is, however, not sufficient as there may be eigenvalues (-1) on the products $[H^{i_1}(X, \mathbb{Q}_l) \times H^{i_2}(C, \mathbb{Q}_l)](\frac{d+1}{2})$ for $i_1 + i_2 = d + 1$, $i_2 \neq 1$.

3.12. Lemma. — *Let $\Phi \in \mathbb{Q}[T]$ be a monic polynomial such that all its roots are roots of unity. Then $|\Phi(-1)|$ is either zero or a power of 2.*

Proof. Φ is a product of cyclotomic polynomials ϕ_n . For these, it is well known [Mo, Section 3] that $\phi_n(-1) = 1$ unless n is a power of 2. Further, the formula $\phi_{2^e}(t) = t^{2^{e-1}} + 1$ shows $\phi_2(-1) = 0$ and $\phi_{2^e}(-1) = 2$ for $e > 1$. Finally, $\Phi_1(-1) = -2$. \square

4 Examples

4.1 The supersingular case

4.1. Proposition. — *Let X be a smooth projective variety of even dimension d over a finite field \mathbb{F}_{p^k} for a prime $p \neq 2$ and k odd. Suppose that $e(X) \equiv 1 \pmod{2}$ and that all eigenvalues of Frob on $H_{\text{ét}}^d(X, \mathbb{Q}_l(d/2))$ are p -adic units. I.e., that the Newton polygon has constant slope $d/2$.*

Then (-1) is an eigenvalue.

Proof. The eigenvalues are l -adic units, too, for every prime number $l \neq p$, hence roots of unity. Theorem 1.9 ensures that $p\Phi(-1)$ is a perfect square. Again, Lemma 3.12 shows that this is possible only for $\Phi(-1) = 0$. \square

4.2. Example. — Let X be a K3 surface over a finite field. Then the Hodge spectral sequence degenerates at E_1 [De3, Proposition 1.1.a)] and, hence, the conjugate spectral sequence degenerates at E_2 [BO, Lemma 8.27.2]. Further, all $H^i(X/W)$ are torsion-free ([Il, II.7.2] or [De3, Proposition 1.1.c)]). Consequently, we have $e(X) = \dim H^2(X, \mathcal{O}_X) = 1$. Cf. Remark 1.8.b).

Assume that the base field is \mathbb{F}_{p^k} for p an odd prime. Theorem 1.9 then asserts that, for z_1, \dots, z_{22} the eigenvalues of Frob on $H_{\text{ét}}^2(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l(1))$, the expression $p^k\Phi(-1) = p^k(1 + z_1) \dots (1 + z_{22})$ is a square in \mathbb{Q} .

In the particular case that X is supersingular and k is odd, Proposition 4.1 shows that (-1) is an eigenvalue. This refines the observation of M. Artin [Ar, 6.8] that the field of definition of the rank-22 Picard group always contains \mathbb{F}_{p^2} .

4.2 Surfaces. The Artin-Tate formula

For surfaces, the assertion of Theorem 1.5 is implied by the Tate conjecture. More precisely,

4.3. Proposition. — *Let X be a smooth projective surface over a finite field \mathbb{F}_q of characteristic p and $\Phi = \Phi_{d/2}^{(d)} \in \mathbb{Q}[T]$ be the characteristic polynomial of Frob on $H_{\text{ét}}^d(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(d/2))$. Put $N := \deg \Phi$ and $\alpha(X) := \dim H^2(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) + \frac{1}{2} \dim H_{\text{ét}}^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. Suppose that the Tate conjecture is true for X .*

Then $(-2)^N q^{\alpha(X)} \Phi(-1)$ is a square in \mathbb{Q} .

Proof. Again, if $\Phi(-1) = 0$ then the assertion is true, trivially. Thus, let us suppose the contrary from now on.

Then Frob and Frob^2 have the eigenvalue 1 with the same multiplicity. In particular, the Tate conjecture predicts the rank of $\text{Pic}(X_{\mathbb{F}_{q^2}})$ not to be higher than that of $\text{Pic}(X)$. Hence, $X_{\mathbb{F}_{q^2}}$, too, fulfills the Tate conjecture.

We are therefore in a situation where the Artin-Tate formula [Mi, Theorem 6.1] computes the discriminants of the Picard lattices $\text{Pic}(X)$ and $\text{Pic}(X_{\mathbb{F}_{q^2}})$, at least up to square factors.

Furthermore, equality of the ranks implies that $\text{disc Pic}(X) / \text{disc Pic}(X_{\mathbb{F}_{q^2}})$ is a necessarily perfect square. This is a standard observation from the theory of lattices. The Artin-Tate formula translates this fact into the assertion that $(-2)^N q^{\alpha(X)} \Phi(-1)$ is a perfect square, cf. [EJ, Lemma 16]. \square

4.4. Remarks. — i) The Artin-Tate formula appears to us as a very natural consequence of the Tate conjecture and the cohomological machinery. Thus, we find it very astonishing that it has the potential to produce incompatible results for a variety and its base extension.

Of course, this does not happen for polynomials that really occur as the characteristic polynomial of the Frobenius on a certain variety. But it occurs for polynomials that otherwise look plausible. This observation was actually the starting point of our investigations.

ii) One might want to compare the Picard lattice of X with that of X_{q^n} for $n > 2$. But this leads to nothing new [EJ, Corollary 18.i)].

iii) Suppose $\dim H^1(X, \mathcal{O}_X) = \frac{1}{2} \dim H_{\text{ét}}^1(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ and that X fulfills the assumptions of Remark 1.8.b). Then $\alpha(X) = e(X) = \dim H^2(X, \mathcal{O}_X)$. We do not know how closely Milne's invariant $\alpha(X)$ and our invariant $e(X)$ are related for “pathological” surfaces.

4.5. Example. — Let X be the double cover of $\mathbf{P}_{\mathbb{F}_7}^2$, given by

$$w^2 = 6x^6 + 6x^5y + 2x^5z + 6x^4y^2 + 5x^4z^2 + 5x^3y^3 + x^2y^4 + 6xy^5 + 5xz^5 + 3y^6 + 5z^6.$$

This is a $K3$ surface of degree two.

The numbers of points on X over the finite fields $\mathbb{F}_7, \dots, \mathbb{F}_{7^{10}}$ are 60, 2 488, 118 587, 5 765 828, 282 498 600, 13 841 656 159, 678 225 676 496, 33 232 936 342 644, 1 628 413 665 268 026, and 79 792 266 679 604 918.

For the characteristic polynomial of Frob on $H_{\text{ét}}^2(X_{\overline{\mathbb{F}}_7}, \mathbb{Q}_l(1))$, this information leaves us with two candidates, one for each sign in the functional equation,

$$\Phi_i(t) = \frac{1}{7} (7t^{22} - 10t^{21} + t^{20} - t^{19} + 6t^{18} - 3t^{17} - 2t^{16} + 4t^{14} - t^{13} - t^{12} + (-1)^i (-t^{10} - t^9 + 4t^8 - 2t^6 - 3t^5 + 6t^4 - t^3 + t^2 - 10t + 7))$$

for $i = 0, 1$. All roots are of absolute value 1.

However, $\Phi_0(-1) = 60/7$ and $(-2)^N 7^e \Phi_0(-1) = 2^{24} \cdot 3 \cdot 5$ is a non-square, which contradicts Theorem 1.5. Therefore, Φ_1 is the characteristic polynomial of Frob on $H_{\text{ét}}^2(X_{\overline{\mathbb{F}}_7}, \mathbb{Q}_l(1))$. The minus sign holds in the functional equation.

4.6. Remark. — Correspondingly, when starting with Φ_0 , the Tate conjecture predicts $\text{rk Pic}(X_{\mathbb{F}_{q^2}}) = \text{rk Pic}(X) = 2$. Further, the Artin-Tate formula states that $\text{disc Pic}(X_{\mathbb{F}_{q^2}}) \in (-465)(\mathbb{Q}^*)^2$ and $\text{disc Pic}(X) \in (-31)(\mathbb{Q}^*)^2$. As $\frac{-465}{-31} = 15$ is a non-square, this is contradictory.

4.3 Cubic fourfolds

4.7. Example. — Let X be the subvariety of $\mathbf{P}_{\mathbb{F}_2}^5$, given by

$$\begin{aligned} & x_0^3 + x_0^2 x_1 + x_0^2 x_4 + x_0^2 x_5 + x_0 x_1 x_2 + x_0 x_1 x_3 + x_0 x_1 x_4 + x_0 x_2 x_3 + x_0 x_2 x_4 \\ & + x_0 x_3 x_4 + x_0 x_4^2 + x_0 x_4 x_5 + x_0 x_5^2 + x_1^3 + x_1 x_2^2 + x_1 x_2 x_4 + x_1 x_2 x_5 + x_1 x_3^2 \\ & + x_1 x_3 x_5 + x_1 x_4^2 + x_1 x_4 x_5 + x_2^3 + x_2^2 x_5 + x_2 x_3^2 + x_2^2 x_4 + x_2^2 x_5 + x_3^2 x_4 + x_3^2 x_5 + x_3 x_4^2 \\ & + x_3 x_5^2 + x_4^3 + x_4^2 x_5 + x_4 x_5^2 + x_5^3 = 0. \end{aligned}$$

This is a smooth cubic fourfold. We have $\dim H^4(X, \mathcal{O}_X) = 0$, $\dim H^3(X, \Omega_X^1) = 1$, and $\dim H^2(X, \Omega_X^2) = 21$. According to Remark 1.8.b), this shows $N = 23$ and $e = 1$.

The numbers of points on X over the finite fields $\mathbb{F}_2, \dots, \mathbb{F}_{2^{11}}$ are 33, 361, 4 545, 69 665, 1 084 673, 17 044 609, 270 543 873, 4 311 990 785, 68 853 026 817, 1 100 586 076 161, and 17 600 769 409 025. The characteristic polynomial of Frob on $H_{\text{ét}}^4(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_l(2))$ is

$$\Phi(t) = \frac{1}{2} (t-1) (2t^{22} - t^{21} - t^{20} + 2t^{19} - 2t^{17} + t^{16} + t^{15} - 2t^{14} + t^{13} + t^{12} - t^{11} + t^{10} + t^9 - 2t^8 + t^7 + t^6 - 2t^5 + 2t^3 - t^2 - t + 2).$$

It turns out that $\Phi(-1) = -1$. The assertion of Theorem 1.9 is true in this example.

4.8. Remark. — The degree 22 factor of Φ is irreducible over \mathbb{Q} . In particular, X is certainly not special in the sense of B. Hassett [Ha].

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